1. Given a set $S \subset \mathbb{N}$, define its upper density by

$$D^*(S) = \limsup_{n\to\infty} \frac{|S \cap \{1, 2, \dots, n\}|}{n}.$$

Suppose that $D^*(S) > 0$. Is it always possible to find $n \in \mathbb{N}$ such that $(S - n) \cap S$ has positive upper density?

2. Let $\mathcal{P}$ be the real linear space of all real polynomials in one variable x and let $L\colon \mathcal{P} \to \mathcal{P}$ be an $\mathbb{R}$-linear mapping. Set $\mathcal{Q} = \cap_{n=1}^{\infty} L^n(\mathcal{P})$. Is it true that $L(\mathcal{Q}) = \mathcal{Q}$?

3. Let $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$ be finite connected graphs (non-oriented, without loops and multiple edges). We say that G_1 and G_2 are *connected-equivalent* if there exists a bijection $f\colon\ E_1 \to E_2$ of their edge sets such that for a subset $Q \subset E_1$ the graph (V_1, Q) is connected if and only if the graph $(V_2, f(Q))$ is connected. Prove that if G_1 and G_2 are connected-equivalent and each graph G_i, $i = 1, 2$, contains a vertex adjacent to every other vertex of G_i, then G_1 and G_2 are isomorphic.

4. Prove for any tetrahedron with vertices A, B, C, D in $\mathbb{R}^3$ one has

$$\frac{AB}{DA + DB} + \frac{BC}{DB + DC} \geq \frac{AC}{DA + DC},$$

where AB is the Euclidean distance between A and B.

5. Let $a_1 \leq a_2 \leq \cdots \leq a_n$ and $b_1 \leq b_2 \leq \cdots \leq b_n$ be real numbers and let $a_i \neq b_j$ for all i, j.

(i) Suppose that a_i and b_i are integer numbers. Prove that the equation

$$e^{a_1 x} + e^{a_2 x} + \cdots + e^{a_n x} = e^{b_1 x} + e^{b_2 x} + \cdots + e^{b_n x}$$

has at most n distinct real roots.

(ii) Prove the assertion in (i) for real a_i and b_i.