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1. A natural number n has prime divisors a and b such that a^2+b^2-1 is divisible by n. Find all such numbers n.

2. We are given a natural number N. A computer picks at random (according to the uniform distribution) a number between 1 and N. Two players are trying to guess this number, making attempts in turn. Each attempt is the following procedure: the player inputs a number between 1 and N that has not been previously input and the computer answers whether the picked number equals the number input by the player or is larger or smaller than it. The winner is the player who inputs the number equal to the picked number. Find the probability that the first player wins provided that both players are trying to maximize their probability to win.

3. Let $ABCDA'B'C'D'$ be a cube of edge length 1. A grasshopper jumps between the lines $A'B$ and $B'C$: from a certain point M_1 on the line $A'B$ it jumps to a point M_2 of the line $B'C$, from there it jumps to a point M_3 of the line $A'B$ ($M_3 \neq M_1$), next, it jumps to a point M_4 of the line $B'C$ ($M_4 \neq M_2$) and so on. The length of each jump is 1. Prove that $M_7 = M_1$.

4. Let $f(x) = \sum_{n=1}^{\infty} x^n/(1+x+x^2+\cdots+x^{n^2})$, $x > 0$. Is the function f continuous on $(0, +\infty)$?

5. Let S_n be the group of bijections f of $\mathbb{N}$ such that $f(x) = x$ for $x > n$. Let $S = \bigcup_{n=1}^{\infty} S_n$. For each permutation π of $\{0, 1, \ldots, n-1\}$ define a bijection a_π of $[0, 1)$ by the equality $na_\pi(x) = \pi([nx]) + \{nx\}$ for $0 \leq x < 1$. (Here $[y]$ and $\{y\}$ denote the integer and fractional parts of y, respectively.) Let R_n denote the set of such bijections (“rearrangements”) of $[0, 1)$ and let $R = \bigcup_{n=1}^{\infty} R_n$. Are the groups S and R isomorphic?

6. Let V be a convex polygon in $\mathbb{R}^2$ of square S and perimeter P. Prove that one can construct a roof over this polygon such that the volume under the roof will be at least $\dfrac{S^2}{2P}$. (A roof is the graph of a nonnegative function on V that is Lipschitz with constant 1 and vanishes on the boundary of V).